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Dynamic realization games in newsvendor inventory centralization

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Abstract Consider a set N of $n(>1)$ stores with single-item and single-period non-deterministic demands like in a classic newsvendor setting with holding and penalty costs only. Assume a risk-pooling single-warehouse centralized inventory ordering option. Allocation of costs in the centralized inventory ordering corresponds to modelling it as a cooperative cost game whose players are the stores. It has been shown that when holding and penalty costs are identical for all subsets of stores, the game based on optimal expected costs has a non empty core (Hartman et al. 2000, Games Econ Behav 31:26–49; Muller et al. 2002, Games Econ Behav 38:118–126). In this paper we examine a related inventory centralization game based on demand realizations that has, in general, an empty core even with identical penalty and holding costs (Hartman and Dror 2005, IIE Trans Scheduling Logistics 37:93–107). We propose a repeated cost allocation scheme for dynamic realization games based on allocation processes introduced by Lehrer (2002a, Int J Game Theor 31:341–351). We prove that the cost subsequences of the dynamic realization game process, based on Lehrer's rules, converge almost surely to either a least square value or the core of the expected game. We extend the above results to more general dynamic cost games and relax the independence hypothesis of the sequence of players' demands at different stages.

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1 Introduction

Consider a situation in which a number of firms or subsidiaries of the same firm undertake a joint project—for instance, centralization of inventory handling facilities. An issue of concern in joint projects is to arrive at a cost allocation acceptable to all parties. Joint ventures that require the resolution of cost (profit) allocation can be effectively addressed within the framework of cooperative game theory. In this paper we examine a well known newsvendor inventory centralization setting.

Suppose we have a set of stores that distribute a single product to their customers. A joint project might consider filling orders from a central facility and shipping directly to the stores' customers—as in catalog stores with sample merchandize. Now suppose that the demand at each of the stores varies randomly with a given distribution function F_i for store i ($i = 1, 2, \ldots, n$) and parameters specific to each store. Each store may independently decide to participate in the centralized ordering arrangement. When participating, the stores would share the costs of the centralized inventory and benefit from the resulted savings.

There have been a number of studies that examined a combined problem of optimizing savings from a centralized inventory and allocating the savings in a manner which maintains the cooperation of participants (Parlar 1988; Hartman and Dror 1996; Anupindi and Bassok 1999; Hartman et al. 2000; Hartman and Dror 2003, 2005; Slikker et al. 2005; Burer and Dror 2007). When logistics providers are setting up inventory and distribution coordination, it is usually referred to as *supply chain management*. In this context Naurus and Anderson (1996), provide a number of enlightening examples of cost cutting when inventory management is coordinated and centralized. If the sharing of benefits is not perceived to be equitable by the firms, the "partnership" may fall apart and the overall benefits might be lost. As Naurus and Anderson (1996) point out, "significant hurdles stand between the idea and its implementation. To begin with, channel members are likely to be skeptical about the rewards of participation...".

The focus of this study is the newsvendor problem, and we start with general remarks regarding the demand distribution and incentives for centralization in an infinitely repeated single period problem.

For general demand distributions $F_i, i = 1, 2, \ldots, n$ and store specific holding and penalty costs there might not be any savings from centralization. Conditions on demand distributions are discussed in Chen and Lin (1989) and on holding and penalty costs in Hartman and Dror (2005). In this study we assume inventory models where the demand at each store is any random variable having null probability of achieving negative values, and allow for correlated stores' demands. We assume identical storage and penalty costs for each store and in the centralized location. Eppen (1979) was the first to show that in this case savings always occur.

The newsvendor inventory centralization problem examined in the literature is geared to the expected value cost analysis. However, minimizing expected centralized inventory cost might not be a very convincing argument for centralization. A build-in cost allocation mechanism should provide additional incentives for cooperation. That is, in each time period the stores reflect on the actual performance of the system in relation to the anticipated long-run expected performance. The analysis of an on-line system cost allocation(s) performance versus the performance in expectation is the main topic of this paper.

The outline of the paper is as follows. Section 2 introduces two games: the newsvendor centralization game referred to as the *newsvendor expected game* and the related *newsvendor realization game*. In addition, another related game, the *dynamic newsvendor realization game* is defined. In Sect. 3 we discuss the repeated allocation process introduced by Lehrer (2002a) and summarize its main findings. In Sect. 4 we apply Lehrer's allocation processes (rules R_1 and R_2, see Lehrer 2002a) to our dynamic realization games to prove the main results in the paper. Specifically, we prove that the diagonal allocation sequence based on Lehrer's rule R_1 applied to the dynamic realization games converges almost surely to some least square value of the expectation game. The other main result states that any accumulation point of the diagonal sequence of allocations based on Leherer's rule R_2 belongs almost surely to the core of the expected game. In Sect. 5 we extend the above results to allocation processes for more general dynamic cost games. We then relax the independence hypothesis of the sequence of players' demands at different stages, replacing it with an appropriate strong stationarity requirement. Similar almost surely convergence results follow.

2 Inventory centralization in newsvendor environments

Suppose a finite set of stores (newsvendors) that respond to a periodic random demand (of newspapers) by ordering a certain quantity at the start of every period. Since the demand is random, in each period a store will face one of two cases: (1) the ordered quantity is less than the realized demand resulting in lost profit for the store; (2) the ordered quantity exceeds the realized demand resulting in a disposal cost for the store since the items (the newspapers) are perishable.

Formally, we consider a set $N = \{1, \ldots, n\}$ of stores. The assumptions of this model are:

1. Let $(\Omega, \mathcal{F}, P)$ be a given probability space. Each store $i \in N$ faces a nonnegative random demand x_i, with distribution function F_i and mean μ_i.
2. The disposal cost is $h > 0$ per unit and lost profit (penalty) cost is $p > 0$ per unit. These costs are the same for all stores and any combination of the stores.
3. The product is ordered once at the start of each period (not reordered), and items on hand at the beginning of the period cannot be returned. There is no order cost and no quantity discounts. Both the demand distributions and the costs are common knowledge. This situation is stationary and infinitely repeated period after period.
4. The cost resulting from an initial inventory of q is

$$\Psi(x, q) = \begin{cases} h(q - x) & \text{if} \quad q \geq x \\ p(x - q) & \text{if} \quad q < x \end{cases}$$

5. Consider a coalition $S \subseteq N$ of stores facing the joint demand $x_S = \sum_{i \in S} x_i$ with distribution function F_S and expected value μ_S. Assume that for all coalitions $S \subseteq N$, $F_S \in £^1(\Omega, \mathcal{F}, P)$. Hence, $E[\Psi(x_S, q)] < \infty$ for all $q \in \mathbb{R}$.

The above is a classic newsvendor setting and for all $S \subseteq N$, we can find a value q_S that minimizes $E[\Psi(x_S, q)]$; i.e., q_S is the optimal order size for S.

We define the *newsvendor expected game* (N, c_E) (henceforth E-game) as the TU cost game with characteristic function $c_E(S) := E[\Psi(x_S, q_S)]$ for all $\emptyset \neq S \subseteq N$. Notice that for $S \subseteq N$, $c_E(S) \geq 0$ represents the optimal expected cost of holding or shortage.

Hartman et al. (2000) proves that the core of E-games is non-empty (i.e., E-games are balanced) for demands with symmetric distribution and for joint multivariate normal demand distribution. Muller et al. (2002) generalizes the result above for all possible joint distributions of the random demands.

Suppose now that in a given period, coalition $S \subseteq N$ decides on an optimal order size q_S. Then, at the end of the period, each player $i \in S$ observes its demand realization, say $\hat{q}_i$. The total demand realization for S is $\hat{q}(S) = \sum_{i \in S} \hat{q}_i$. Just as for a single store there are two possibilities: (1) $\hat{q}(S) \leq q_S$, and the cost for this centralized system is $h(q_S - \hat{q}(S))$; (2) $\hat{q}(S) \geq q_S$ and the cost is equal to $p(\hat{q}(S) - q_S)$.

The *newsvendor realization game* (henceforth R-game), (N, c_R), is defined by $c_R(S) := \max\left\{p(\hat{q}(S) - q_S), h(q_S - \hat{q}(S))\right\}$ for all $\emptyset \neq S \subseteq N$, where q_S is the demand of S in the E-game. This non-negative game measures the actual cost of the demand realization for every $S \subseteq N$. Hartman and Dror (2005) shows that R-games are not balanced in general (i.e., the core may be empty), by providing a realization example for joint multivariate normal demand distribution.

The reader may note that E-games and R-games are related by means of a long-term expectation property: $E[c_R(S)] = E\left[\Psi(x, \hat{q}(S))\right] = c_E(S)$ for all $\emptyset \neq S \subseteq N$. The above property means that the long-term average cost of coalition S, for repeated realizations of the actual cost game c_R, is the same as its cost in the expected cost game c_E, provided that the underlying individual demand distributions do not change.

To complete this section we introduce a dynamic analysis of the newsvendor situation where we take into account the repeated approach of newsvendor realization games.

Consider the stores over any finite horizon T (T is a positive integer counting the number of single periods in the finite horizon). In every period t we observe actual demands quantities $\hat{q}_i^t$, for all $i = 1, \ldots, n$. For a fixed t, those are realizations of the demand random variables. The sequence of demand realizations faced by store i, $\left\{\hat{q}_i^t\right\}_{t \geq 1}$, is ruled by the distribution function F_i, and we assume that $\hat{q}_i^t$ and $\hat{q}_i^{t'}$ are independent for any $t \neq t'$. Hence $\hat{q}_i^t$ and $\hat{q}_i^{t'}$ are independent and identically distributed (i.i.d.) random variables for all $t \neq t'$.

Consider, for each store i, $i = 1, \ldots, n$, the average sequence of demand realizations $\left\{\tilde{q}_i^T\right\}_{T \geq 1}$, where $\tilde{q}_i^T := \frac{1}{T} \sum_{t=1}^{T} \hat{q}_i^t$.

We define the *dynamic newsvendor realization game* (DR-game) at stage T, $(N, \tilde{c}_R^T)$, by the following characteristic function:

$$\tilde{c}_R^T(S) := \max\{p(\tilde{q}^T(S) - q_S), h(q_S - \tilde{q}^T(S))\}, \tag{1}$$

for all $\emptyset \neq S \subseteq N$, where $\tilde{q}^T(S) = \sum_{i \in S} \tilde{q}_i^T$.

Given a sequence of actual realizations of demand $\hat{q} = \{\hat{q}^T\}_{T \geq 1}$ with $\hat{q}^T = (\hat{q}_i^T)_{i=1,\ldots,n}$, DR -games $(N, \tilde{c}_R^T(\hat{q}))_{T \geq 1}$ are standard TU cost cooperative games.

The stochastic nature of the DR-games $(N, \tilde{c}_R^T(\hat{q}))_{T \geq 1}$ is ruled by the random demands from where we draw the $(\hat{q})$ realizations and it is described through the sequences of random variables $\{\tilde{c}_R^T(S)\}_{T \geq 1}$ for all $S \subseteq N$. Note that the probability distribution function of the random variable $\tilde{c}_R^T(S)$ is given by:

$$P\left[\tilde{c}_R^T(S) \leq r\right] = P\left[\begin{array}{l}\text{all the outcomes of } \hat{q}_i^t, i = 1, \ldots, n, t = 1, \ldots, T \text{such that} \\ \max\{p(\tilde{q}^T(S) - q_S), h(q_S - \tilde{q}^T(S))\} \leq r\end{array}\right],$$

for all $r \in \mathbb{R}$. Hence, the class of DR-games is a subclass of the class of TU cooperative games with random worths. Notice that for any $\hat{q}$, the DR-game at stage $T = 1$ coincides with the R-game; i.e., $\tilde{c}_R^1(\hat{q})(S) = c_R(S)$ for all $S \subseteq N$.

There are few models of cooperative games where the worth of a coalition may be uncertain; these games are called stochastic cooperative games. For a clear and detailed overview see Suijs (2000) (see also Granot 1977). Fernandez et al. (2002) introduce cooperative games with random payoffs. The random payoffs are compared by means of stochastic orders. Timmer et al. (2003, 2005) and Timmer (2006) study a model where the stochastic value of coalitions depends on a set of actions that every coalition can take. Several solution concepts (Core, Shapley-like, compromise value) for all the above stochastic games have been analyzed.

In this paper, we focus on solutions for DR-games using dynamic (time dependent) processes. Our analysis builds upon the work of Lehrer (2002a,b).

3 Repeated allocation processes

In real-life we may expect players to monitor their costs of inventory centralization one period at a time and, accordingly, form an " opinion/response" regarding the fairness of their cost allocations. From now on, we focus on a repeated allocation process.

Lehrer (2002a) describes four allocation rules in a stylized cooperative game repeated an infinite number of times. At each time period the same game is played with a fixed budget of size B that has to be distributed among the players in a finite set N. The game has its characteristic function v with the interpretation that $v(S)$ represents the needs of coalition $S \subseteq N$. The allocation at period t, $t = 0, 1, 2, \ldots$ is a vector $a_t = (a_t^i)$, where a_t^i is the portion of the budget B allocated to player $i \in N$ at time t. An allocation rule determines the allocation at time T, a_T, as a function of v and of all a_t, $t < T$. The sequence $(a_t)_{t \geq 1}$ is the allocation process induced by the rule.

The empirical distribution of the budget among the players is, at any stage, an allocation of the budget. We focus on the first two types of processes introduced by Lehrer, to be implemented later on the DR -games at each stage T (see next section). The first type of process is based on the idea that giving a budget to a player increases the total well-being of the entire group. The player whose marginal contribution to this well-being is maximal will receive the budget. It is shown that any process of this type generates allocations that converge to some least square value (introduced by Ruiz et al. 1998) The second type of allocation process is defined inductively and a player whose weighted actual surplus is nonnegative is chosen and given the entire budget. This process converges either to the core of the game, when the game is balanced, or to the least core. Convergence in this context means that the distance between the core (or the least core) and the empirical sequence of allocations shrinks to zero. The proofs of the above two types of allocation processes rely on a geometric principle that lies behind Blackwell's approachability result (Blackwell 1956; Lehrer 2002b).

Formally, let N be a finite set of players, where the number of players, $|N|$, is $n > 1$. Consider a normalized cooperative game v (i.e., $v(N) = 1$). Let $A = \{(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n : \sum_{i=1}^{n} a_i = v(N) = 1 \text{ and } a_i \geq v(\{i\}) \text{for all} i \in N\}$ be the set of allocations.

An allocation rule R is a function $R : \cup_{t=0}^{\infty} A^t \to A$, where A^t is the Cartesian product of A with itself t times and A^0 is a singleton that represents the empty history of allocations.

Any allocation rule induces a sequence $a_1, a_2, \ldots$ of allocations in A as follows: a_1 is the first allocation R prescribes, $a_2 = R(a_1)$, $a_3 = R(a_1, a_2)$, etc. This sequence is called the allocation sequence induced by R. For any time t, denote by $\overline{a}_t$ the historical distribution of the budget up to time t. That is, $\overline{a}_t^i$ is the frequency of the stages up to time t, where player $i \in N$ received the budget. For any $S \subseteq N$, let $\overline{a}_t(S)$ be $\sum_{i \in S} \overline{a}_t^i$.

Below we summarize the first two of Lehrer's allocation rules by means of the corresponding allocation process.

1. *Processes that converge to the least square value (R_1 allocation rule)*
   A coalition is chosen randomly according to the probability distribution $(\alpha_S)_{S \subseteq N}$. At time $t + 1$ the coalition S is assigned a weight proportional to the excess corresponding to the allocation $\overline{a}_t$, $\overline{a}_t(S) - v(S)$. At any time a player whose contribution to the expected weighted welfare of society $\sum_{S \subseteq N} \alpha_S \left(\overline{a}_t(S) - v(S)\right) \left(\mathbb{I}_{i \in S} - v(S)\right)$, is maximal, is chosen (where $\mathbb{I}_{i \in S} - v(S)$ is $1 - v(S)$ if $i \in S$ and $-v(S)$ otherwise). This player receives the entire budget and is denoted player i_{t+1}.
   Let R_1 be the allocation rule induced by the above process (see Lehrer 2002a for further details). The following theorem shows that this rule generates an allocation sequence that converges to some least square value.

   Theorem 3.1 (Lehrer 2002a). *Let $a_1, a_2, \ldots$ be the allocation sequence induced by R_1. Then, $\overline{a}_t$ converges to the least square value of the game that corresponds to the weights α_S, $S \subseteq N$.*

2. *Processes that converge to the core and to the least core (R_2 allocation rule)* Let v be a balanced game. Let $S_1, \ldots, S_k$, $k = 2^n - 1$, be the list of all non-empty coalitions of N. Denote by y_i the vector in $\mathbb{R}^k$ whose lth coordinate is $\mathbb{I}_{i \in S_l} - v(S_l)$. Two sequences are defined: the allocation process $a_1, a_2, \ldots$ of vectors in $\mathbb{R}^n$, by means of vectors of the standard basis of $\mathbb{R}^n$, and an auxiliary sequence $z_1, z_2, \ldots$ of vectors in $\mathbb{R}^k$, by means of vectors y_i in $\mathbb{R}^k$. The number $\overline{z}_t^l$ measures the historical average surplus of the coalition S_l up to stage t. At this stage the coalitions are weighted with respect to these surpluses: those coalitions with a positive surplus

are neglected while the other coalitions are assigned weights proportional to their (negative) surplus (i.e., for such a coalition, say, S_l, the weight is $-\min(\overline{z}_t^l, 0)$). Then, a player i whose weighted actual surplus (i.e., $\sum_{l=1}^{k}[-\min(\overline{z}_t^l, 0)]$ $[\mathbb{I}_{i\in S_l} - v(S_l)]$ is non-negative is chosen and is given the entire budget $v(N)$.
Let R_2 be the allocation rule induced by the above process. The following theorem shows that any limit point of the corresponding allocation sequence $\overline{a}_t$ is in the core.

Theorem 3.2 (Lehrer 2002a). *Let $a_1, a_2, \ldots$ be the allocation sequence induced by R_2. Then, $\overline{a}_t$ converges to the core of the game. That is, any accumulation point of the sequence $\overline{a}_t$ is in the core.*

Notice that in constructing the allocation process that converges to the core we assumed that the game is balanced. As noted in Lehrer (2002a), rule R_2 can be modified in the case of an empty core to obtain an allocation sequence which converges to a point in the least core (the intersection of all non-empty ϵ-cores).

4 Allocation processes for the dynamic newsvendor realization game

In this section, we prove that there exist Lehrer's allocation processes (rules R_1 or R_2) applied to the DR-games at each stage T, that converge almost surely (that is, the set of outcomes where it does not converges has null probability) either to some least square value or to the core of E-games.

We start proving a technical lemma that will be useful in our analysis. Previously, given any TU game (N, c), we recall that the least square value (see Ruiz et al. 1998) for a weight function $\alpha = (\alpha_S)_{S\subseteq N}$ is:

$$LS^\alpha(c) = (LS_1^\alpha(c), \ldots, LS_n^\alpha(c)),$$

where

$$LS_i^\alpha(c) = \frac{c(N)}{n} + \frac{1}{n\beta}\left[\sum_{S:i\in S}(n-s)\alpha(S)c(S) - \sum_{S:i\notin S} s\alpha(S)c(S)\right]$$

and $\beta = \sum_{s=1}^{n-1}\alpha(S)\binom{n-2}{s-1}$.

Notice that the least square value for a weight function α can easily be extended to stochastic cooperative games; in particular to DR-games. Indeed, take the continuous functions $g : \mathbb{R} \longrightarrow \mathbb{R}$ and $f_i : \mathbb{R}^{2^n-1} \longrightarrow \mathbb{R}$ defined by

$$g(y) = \max\{p(y - q_S), h(q_S - y)\},$$

$$f_i(x) = \frac{g(x_N)}{n} + \frac{1}{n\beta}\left[\sum_{S:i\in S}(n-s)\alpha(S)g(x_S) - \sum_{S:i\notin S} s\alpha(S)g(x_S)\right],$$

for all $y \in \mathbb{R}$ and $x = (x_S)_{S \subseteq N} \in \mathbb{R}^{2^n - 1}$. Then, for all $i = 1, \ldots, n$ and all $T \geq 1$,

$$LS_i^{\alpha}\left(\tilde{c}_R^T\right) = f_i\left(\left(\tilde{c}_R^T(S)\right)_{S \subseteq N}\right) \tag{2}$$

is a random variable with values $LS_i^{\alpha}\left(\tilde{c}_R^T(\hat{q})\right) = f_i\left(\left(\tilde{c}_R^T(\hat{q})(S)\right)_{S \subseteq N}\right)$, for any sequence of actual realizations of demand $\hat{q}$.

Lemma 4.1 *For any $S \subseteq N$ and any weight function α (that does not depend on T),*

$$\tilde{c}_R^T(S) \xrightarrow[T \to \infty]{a.s.} c_E(S). \tag{3}$$

$$LS_i^{\alpha}\left(\tilde{c}_R^T\right) \xrightarrow[T \to \infty]{a.s.} LS_i^{\alpha}(c_E) \quad \forall i \in N. \tag{4}$$

Proof Consider the continuous function g defined above. Recall that $\hat{q}^t(S) = \sum_{i \in S} \hat{q}_i^t$ and consider the sequence $\left\{\hat{q}^t(S)\right\}_{t \geq 1}$. It is clear that the random variables in the sequence are i.i.d. with mean value μ_S. Therefore, by the Strong Law of Large Numbers,

$$\tilde{q}^T(S) = \frac{1}{T}\sum_{t=1}^{T} \hat{q}^t(S) \xrightarrow[T \to \infty]{a.s.} \mu_S.$$

Hence, by the continuity of g,

$$g\left(\tilde{q}^T(S)\right) = \tilde{c}_R^T(S) \xrightarrow[T \to \infty]{a.s.} g(\mu_S) = c_E(S)$$

which proves (3). Applying (3) we have

$$LS_i^{\alpha}\left(\tilde{c}_R^T\right) = f_i\left(\left(\tilde{c}_R^T(S)\right)_{S \subseteq N}\right) \xrightarrow[T \to \infty]{a.s.} f_i((c_E(S))_{S \subseteq N}) = LS_i^{\alpha}(c_E).$$

This proves (4). □

Our approximation process consists of applying Lehrer's allocation rules R_1 and R_2 (described in Sect. 3) at each stage T to each one of our DR-games. We then examine for the repeated realization process the rules' properties.

Assume that we are given a generic game (N, c). Let $\{a^{R_i}(c)(l)\}_{l \geq 1}$ denote Lehrer's allocation sequence induced by R_i $i = 1, 2$. (By $a^{R_i}(c)(l)$ we refer to the lth element in the corresponding sequence.) Let $\{\bar{a}^{R_1}(c)(l)\}_{l \geq 1}$ and $\{\bar{a}^{R_2}(c)(l)\}_{l \geq 1}$ be the allocation schemes converging to some least square value with weight function α (the former) and converging to an element of the least core (the latter).

Notice that Lehrer's allocation sequence induced by R_i $i = 1, 2$, can also be extended to stochastic cooperative games, in particular to DR-games. Indeed, Lehrer's allocation scheme applied to a DR-game $\left(N, \tilde{c}_R^T\right)$ is a random variable $\bar{a}^{R_i}\left(\tilde{c}_R^T\right)$, with values $\bar{a}^{R_i}\left(\tilde{c}_R^T(\hat{q})\right)$ for any sequence of actual realizations of demand $\hat{q}$.

Our first theorem ensures the convergence of Lehrer's $\{\bar{a}^{R_1}(c)(l)\}_{l\geq 1}$ allocation scheme, applied to DR-games, to some least square value of E-games. In this case, we obtain that for any sequence of actual demand realizations the sequence of diagonal steps $\{\bar{a}^{R_1}\left(\tilde{c}_R^T\right)(T)\}$ (i.e. the one that chooses the T-th replication of Lehrer's R_1-approach at stage T of the realization game) converges almost surely to some least square value of the E-games.

Theorem 4.2 *Let (N, c_E) be a newsvendor expected game. The diagonal sequence $\{\bar{a}^{R_1}\left(\tilde{c}_R^T\right)(T)\}_{T\geq 1}$ converges almost surely to $LS^\alpha(c_E)$.*

Proof First of all, we note that according to Theorem 3.1, for any sequence of actual realizations of demand $\hat{q}$,

$$\bar{a}^{R_1}\left(\tilde{c}_R^T(\hat{q})\right)(l) \underset{l\to\infty}{\overset{\text{pointwise}}{\longrightarrow}} LS^\alpha\left(\tilde{c}_R^T(\hat{q})\right), \quad \forall T \geq 1.$$

Therefore, applying the pointwise convergence of the above process together with (4) we have the diagram:

$$\begin{array}{ccc} LS^\alpha\left(\tilde{c}_R^T\right) & \underset{l\to\infty}{\overset{\text{pointwise}}{\longleftarrow}} & \bar{a}^{R_1}\left(\tilde{c}_R^T\right)(l) \\ {\scriptstyle T\to\infty}\Big\downarrow{\scriptstyle a.s.} & & \\ LS^\alpha(c_E). & & \end{array}$$

Hence, taking any infinite subsequence with increasing indexes in (l, T) we obtain almost sure convergence to $LS^\alpha(c_E)$. In particular, following the diagonal sequence, namely taking indexes (T, T), $T \geq 1$, we get the result in the theorem. □

Our next result explains the approachability of $\left\{\bar{a}^{R_2}(c)(l)\right\}_{l\geq 1}$ allocation scheme, applied to DR-games, to the core of E-games. In this case, we prove that any accumulation point of the sequence of diagonal steps $\left\{\bar{a}^{R_2}\left(\tilde{c}_R^T\right)(T)\right\}$ converges almost surely to a point in the core of E-games.

The core core(c_E) of the game (c_E, N) is:

$$\text{core}(c_E) = \left\{x \in \mathbb{R}^n : \sum_{i\in S} x_i \leq c_E(S),\ \forall S \subset N,\ \sum_{i\in N} x_i = c_E(N)\right\}.$$

The same definition is applicable to the core of the realization game defined on a sequence of actual realizations of demand $\hat{q} = \left\{\hat{q}^t\right\}_{t\leq T}$ at any stage T; i.e. core $\left(\tilde{c}_R^T(\hat{q})\right)$. Analogously, the ε-core of that game for any $\varepsilon > 0$ is:

$$\text{core}\left(\tilde{c}_R^T(\hat{q}), \varepsilon\right) = \left\{x \in \mathbb{R}^n : x(S) \leq \tilde{c}_R^T(\hat{q})(S) + \varepsilon,\ S \subset N;\ x(N) = \tilde{c}_R^T(\hat{q})(N)\right\},$$

and the least core is:

$$L\text{core}\left(\tilde{c}_R^T(\hat{q})\right) = \bigcap_{\text{core}\left(\tilde{c}_R^T(\hat{q}),\varepsilon\right)\neq\emptyset} \text{core}\left(\tilde{c}_R^T(\hat{q}), \varepsilon\right).$$

However, how to define core $\left(\tilde{c}_R^T\right)$ and core $\left(\tilde{c}_R^T, \varepsilon\right)$ is not clear since $\tilde{c}_R^T$ is a stochastic cooperative game at each stage T. Therefore, we must first define these sets.

First, we have to extend the concept of efficiency. Note that when $\tilde{c}_R^T(N)$ is an absolutely continuous random variable, the event $\tilde{c}_R^T(N) = x(N)$ has null probability. Thus, the induced concept of core would be a set with null probability. To overcome this difficulty, we define efficiency through a significance level around the average value of the random variable.

Given a random variable Y with $E(Y) = \bar{y}$ and a significance level β, $0 \leq \beta \leq 1$, let $\phi(\beta) = \inf\{\phi' : P[|Y - \bar{y}| \leq \phi'] \geq \beta\}$. We say that a vector x is $\phi(\beta)$-efficient if $|x(N) - \bar{y}| \leq \phi(\beta)$.

In our setting core $\left(\tilde{c}_R^T\right)$ and core $\left(\tilde{c}_R^T, \varepsilon\right)$ are random sets. Therefore, to define these sets we have to state the significance level β_T of that efficiency, which in turns induces the values $\phi_T(\beta_T)$. For simplicity, we denote those values $\phi_T(\beta_T)$ as ϕ_T, when no confusion is possible. Then, the probability of an allocation x to be in those cores is given as:

$$P\left[x \in \text{core}\left(\tilde{c}_R^T\right)\right] = P\left[x(S) \leq \tilde{c}_R^T(S), \forall S \subset N; \left|x(N) - E\left(\tilde{c}_R^T(N)\right)\right| \leq \phi_T\right], \tag{5}$$

$$P\left[x \in \text{core}\left(\tilde{c}_R^T, \varepsilon\right)\right] = P\left[x(S) \leq \tilde{c}_R^T(S) + \varepsilon, \forall S \subset N; \left|x(N) - E\left(\tilde{c}_R^T(N)\right)\right| \leq \phi_T\right]. \tag{6}$$

Note that setting $\beta_T = 1$ for all T, provided that ϕ_T is not identically equal to $+\infty$, we have by (3) that $\phi_T \underset{T\to\infty}{\to} 0$.

Then, $x \in$ core $\left(\tilde{c}_R^T\right)$ or core $\left(\tilde{c}_R^T, \varepsilon\right)$ almost surely if and only if $P\left[x \in \text{core}\left(\tilde{c}_R^T\right)\right] = 1$ or $P\left[x \in \text{core}\left(\tilde{c}_R^T, \varepsilon\right)\right] = 1$, respectively.

Theorem 4.3 *Let (N, c_E) be a newsvendor expected game. Then any accumulation point of the sequence $\left\{\bar{a}^{R_2}\left(\tilde{c}_R^T\right)(T)\right\}_{T\geq 1}$ belongs to $core(c_E)$ almost surely.*

Proof First, we prove that there exist $\widehat{x} \in \mathbb{R}^n$ such that $\left|\widehat{x}(N) - E\left(\tilde{c}_R^T(N)\right)\right| \leq \phi_T$ and a sequence $\{\varepsilon_T\}_{T\geq 1}$, which converges almost surely to zero, such that $\widehat{x} \in$ core $\left(\tilde{c}_R^T, \varepsilon_T\right)$ a.s.

Define $\varepsilon_T(S) := c_E(S) - \tilde{c}_R^T(S)$, for all $S \subset N$ and let $\{\phi_T\}_T$ any sequence converging to 0. By Lemma 4.1, $\varepsilon_T(S) \underset{T\to\infty}{\xrightarrow{a.s.}} 0$ for all $S \subset N$. Let $\varepsilon_T := \max_{S\subset N}\{|\varepsilon_T(S)|\}$ for all $T \geq 1$. Then, for all $\delta > 0$, there exists $T(\delta)$ such that for all $T > T(\delta)$, $\varepsilon_T < \delta$ almost surely.

Suppose that for all $x \in \mathbb{R}^n$, such that $\left|x(N) - E\left(\tilde{c}_R^T(N)\right)\right| \leq \phi_T$ (ϕ_T induced by the significance level β_T), $P\left[x \notin \text{core}\left(\tilde{c}_R^T, \varepsilon_T\right)\right] > 0$. Then,

$$P\left[x(S) > \tilde{c}_R^T(S) + \varepsilon_T, \text{ for some } S \subset N\right] > 0. \tag{7}$$

Now, since for any $S \subset N$, the condition $x(S) > \tilde{c}_R^T(S) + \varepsilon_T$ implies $x(S) > c_E(S)$, we obtain by (7) that

$$P\left[x(S) > c_E(S), \text{ for some } S \subset N\right] > 0,$$

which is a contradiction, since taking $x^* \in \text{core}(c_E)$,

$$P[x^*(S) > c_E(S), \text{ for some } S \subset N] = 0.$$

Hence, we conclude that there exists $\widehat{x} \in \mathbb{R}^n$ such that $\left|\widehat{x}(N) - E\left(\tilde{c}_R^T(N)\right)\right| \leq \phi_T$ satisfying $P\left[\widehat{x} \in \text{core}\left(\tilde{c}_R^T, \varepsilon_T\right)\right] = 1$.

Next, we prove that for any realization of actual demands at stage T, $\hat{q}^1, \ldots, \hat{q}^T$, any accumulation point x^T of Lehrer's R_2 repeated allocation process, constructed on the game $\left(N, \tilde{c}_R^T(\hat{q})\right)$, satisfies $P\left[x^T \in \text{core}\left(\tilde{c}_R^T, \varepsilon_T\right)\right] = 1$.

Indeed,

$$\begin{aligned} &P\left[x^T \in \text{core}\left(\tilde{c}_R^T, \varepsilon_T\right)\right] \\ &\quad = P\left[\left\{\hat{q} \middle/ \begin{array}{l} x^T(\hat{q})(S) \leq \tilde{c}_R^T(\hat{q})(S) + \varepsilon_T(\hat{q}), \forall S \subset N; \\ \left|x^T(\hat{q})(N) - E\left(\tilde{c}_R^T(\hat{q})(N)\right)\right| \leq \phi_T \end{array}\right\}\right] = 1, \end{aligned}$$

since Theorem 3.2 ensures that for any realization $\hat{q}$, $x^T(\hat{q}) \in \text{core}\left(\tilde{c}_R^T(\hat{q}), \varepsilon_T(\hat{q})\right)$ provided that this set is not empty.

Next, we prove that for any accumulation point x of a sequence $\left\{x^T\right\}_{T\geq 1}$ such that $x^T \in \text{core}\left(\tilde{c}_R^T, \varepsilon_T\right)$ a.s. for all $T > 1$, then $\mathrm{x} \in \text{core}(c_E)$ almost surely.

By Lemma 4.1, $\tilde{c}_R^T(S) \xrightarrow[T\to\infty]{a.s.} c_E(S)$ for all S. Thus, for any $\delta > 0$ small enough, there exists $T(\delta)$ such that for all $T > T(\delta)$: $\tilde{c}_R^T(S) < c_E(S) + \delta$ almost surely. Since $x^T \in \text{core}\left(\tilde{c}_R^T, \varepsilon_T\right)$ almost surely, it follows that:

$$\begin{aligned} &x^T(S) \leq \tilde{c}_R^T(S) + \varepsilon_T < c_E(S) + \delta + \varepsilon_T, \text{ a.s. and for all } S \subset N, \\ &\phi_T \geq \left|x^T(N) - E\left(\tilde{c}_R^T(N)\right)\right|, \end{aligned}$$

for all $T > T(\delta)$.

Hence, as $\delta \to 0$ we have that $T(\delta) \to \infty$, and the accumulation point x satisfies:

$$\mathrm{x}(S) \leq c_E(S) \text{ for all } S \subset N, \text{ a.s. and } |\mathrm{x}(N) - c_E(N)| \leq 0, \tag{8}$$

since $\varepsilon_T(S) \xrightarrow[T\to\infty]{a.s.} 0$ for all $S \subseteq N$ and $\phi_T \xrightarrow[T\to\infty]{} 0$.

Thus, any accumulation point of the sequence $\left\{\bar{a}^{R_2}\left(\tilde{c}_R^T\right)(T)\right\}_{T\geq 1}$ belongs to $\text{core}\left(\tilde{c}_R^T, \varepsilon_T\right)$ almost surely for all $T > 1$. Hence, applying (8), any accumulation point of $\left\{\bar{a}^{R_2}\left(\tilde{c}_R^T\right)(T)\right\}$ belongs to $\text{core}(c_E)$ almost surely. □

5 Final comments and remarks

5.1 Allocation processes for dynamic cost games

In the previous section we present two allocation processes for dynamic newsvendor realization games. Each of them was based on applying Lehrer's allocation processes induced by rules R_1 and R_2, respectively, to DR-games at each stage $T \geq 1$.

In this subsection we extend all the above results to a more general framework. We state two allocation processes for general dynamic cost games, which are also based on applying Lehrer's allocation processes (rules R_1 or R_2).

Let (N, c) be a non-negative balanced TU cost game. Let $\{(N, c^t)\}_{t\geq 1}$ (t is no longer an index for a time period like in a newsvendor game but simply an index of a game in a sequence) be a sequence of non-negative stochastic cooperative games such that

$$c^t(S) \xrightarrow[t\to\infty]{a.s.} c(S), \text{ for any } S \subseteq N.$$

It is clear that for a given scenario $\hat{q}$ of the sequence of stochastic cooperative games (that is, a realization of the stochastic behavior), $(N, c^t(\hat{q}))_{t\geq 1}$ are standard TU cost games. Therefore, we can apply Lehrer's R_i-allocation schemes, $i = 1, 2$, to each of them.

Then we can obtain similar results to those in Theorems 4.2, and 4.3.

Theorem 5.1 *Let (N, c) be a non-negative balanced TU cost game and $\left\{(N, c^t)\right\}_{t\geq 1}$ a sequence of non-negative cooperative games satisfying that $c^t(S) \xrightarrow{a.s.} c(S)$, for all $S \subset N$. Then, the diagonal sequence $\left\{\bar{a}^{R_1}(c^t)(t)\right\}_{t\geq 1}$ satisfies:*

$$\bar{a}^{R_1}(c^t)(t) \xrightarrow[t\to\infty]{a.s.} LS^\alpha(c).$$

Theorem 5.2 *Let (N, c) be a non-negative balanced TU cost game and $\left\{(N, c^t)\right\}_{t\geq 1}$ a sequence of non-negative stochastic cooperative games satisfying that $c^t(S) \xrightarrow{a.s.} c(S)$, for all $S \subset N$. Then, any accumulation point of the sequence $\left\{\bar{a}^{R_2}(c^t)(t)\right\}_{t\geq 1}$ belongs to $core(c)$ almost surely.*

5.2 Removing the independence hypothesis

The approachability results for the DR-games can be further extended removing the independence hypothesis of the sequence of players' demand at different stages. Instead, we will require the *stochastic processes* $\{\hat{q}_i^t\}_{t\geq 1}$ to be strongly stationary for any $i \in N$. Under this hypothesis the sequences $\{\hat{q}^t(S)\}_{t\geq 1}$ inherit the same character (strongly stationary) and by the *ergodic theorem* (see Feller 1966) the sequences $\{\tilde{q}^T(S)\}_{T\geq 1}$, where $\tilde{q}^T(S) = \frac{1}{T}\sum_{t=1}^{T}\hat{q}^t(S)$, converge almost surely to a random variable, Y_S, satisfying $E[Y_S] = \mu_S$ for any $S \subseteq N$.

Under this hypothesis one can extend, *mutatis mutandis*, Lemma 4.1 as follows:

Lemma 5.3 *For any $S \subseteq N$ and any weight function α (which does not depend on T),*

$$\tilde{c}_R^T(S) \xrightarrow[T\to\infty]{a.s.} Y_S \quad \text{where} \quad E[Y_S] = c_E(S). \tag{9}$$

$$LS_i^\alpha\left(\tilde{c}_R^T\right) \xrightarrow[T\to\infty]{a.s.} Y_{LS_i} \quad \text{where} \quad E[Y_{LS_i}] = LS_i^\alpha(c_E) \quad \forall i \in N. \tag{10}$$

Using this lemma we get similar results to Theorem 4.2.

Theorem 5.4 *Let (N, c_E) be a newsvendor expected game. The diagonal sequence $\left\{\bar{a}^{R_1}\left(\tilde{c}_R^T\right)(T)\right\}_{T\geq 1}$ satisfies:*

$$\bar{a}^{R_1}\left(\tilde{c}_R^T\right)(T) \xrightarrow[T\to\infty]{a.s.} Y_{LS_i}, \quad \text{where} \quad E[Y_{LS_i}] = LS_i^\alpha(c_E), \quad \forall i = 1, \ldots, n.$$

The extension of Theorem 4.3 seems to be more involved and needs further investigation. The main difference is that the sequence of characteristic functions converges now to a random vector $Y = \{Y_S\}_{S\subseteq N}$ and therefore the limit defines a random set core(Y). The probability of an allocation to belong to core(Y) is given by:

$$P[x \in \text{core}(Y)] = P\left[x(S) \leq Y_S, \forall S \subset N; |x(N) - E(Y_N)| \leq \phi_Y(\beta)\right], \tag{11}$$

for a significance level of efficiency β. Analogously, we introduce core(Y, ε) as the random set defined by

$$P[x \in \text{core}(Y, \varepsilon)] = P\left[x(S) \leq Y_S + \varepsilon,\ \forall S \subset N; |x(N) - E(Y_N)| \leq \phi_Y(\beta)\right], \tag{12}$$

where

$$\phi_Y(\beta) := \inf\left\{\delta \,\middle/\, P[|E(Y_N) - Y_N| \leq \delta] \geq \beta\right\}.$$

Theorem 5.5 *Suppose that $Y_S \geq 0$ for all $S \subseteq N$, $\phi_Y(1) < +\infty$ and for some $T' > 1$ there exists $x \in$ core(Y) almost surely satisfying $\left|x(N) - E\left(\tilde{c}_R^T(N)\right)\right| \leq \phi_T$ for all $T \geq T'$. Then, any accumulation point $\bar{x}$ of the sequence $\left\{\bar{a}^{R_2}\left(\tilde{c}_R^T\right)(T)\right\}_{T\geq 1}$ belongs to $core(Y)$ almost surely.*

Proof First, we prove that there exist $\widehat{x} \in \mathbb{R}^n$ such that $\left|\widehat{x}(N) - E\left(\tilde{c}_R^T(N)\right)\right| \leq \phi_T$ and a sequence $\{\varepsilon_T\}_{T\geq 1}$ which converges almost surely to zero such that $P\left[\widehat{x} \in \text{core}\left(\tilde{c}_R^T, \varepsilon_T\right)\right] = 1$.

Define $\varepsilon_T(S) := Y_S - \tilde{c}_R^T(S)$, for all $S \subset N$ and let $\{\phi_T\}_T$ be a sequence converging to 0. By Lemma 5.3, $\varepsilon_T(S) \xrightarrow[T\to\infty]{a.s.} 0$ for all $S \subseteq N$. Set $\varepsilon_T := \max_{S\subseteq N}\{|\varepsilon_T(S)|\}$, for all $T \geq 1$. Then, for all $\delta > 0$, $\exists T(\delta)$ such that for all $T > T(\delta)$, $\varepsilon_T < \delta$ almost surely.

Suppose that for all $x \in \mathbb{R}^n$ such that $\left|x(N) - E\left(\tilde{c}_R^T(N)\right)\right| \leq \phi_T$ (being ϕ_T the threshold induced by the efficiency level β_T), $P\left[x \notin \text{core}\left(\tilde{c}_R^T, \varepsilon_T\right)\right] > 0$.

Then

$$P\left[x(S) > \tilde{c}_R^T(S) + \varepsilon_T,\ \text{for some } S \subset N\right] > 0. \tag{13}$$

Now taking into account that $x(S) > \tilde{c}_R^T(S) + \varepsilon_T$ a.s. implies $x(S) > Y_S$ a.s., we obtain by (13) that

$$P\left[x(S) > Y_S,\ \text{for some } S \subset N\right] > 0,$$

which is a contradiction, since taking $x^* \in \text{core}(Y)$ a.s. satisfying $\left|x^*(N) - E\left(\tilde{c}_R^T(N)\right)\right| \le \phi_T$ for all $T \ge T'$, we have

$$P[x^*(S) > Y_S,\ \text{for some } S \subset N] = 0.$$

Hence, there exists $\widehat{x} \in \mathbb{R}^n$ such that $P[\widehat{x} \in core(\tilde{c}_R^T, \varepsilon_T)] = 1$ for all $T \ge T'$.

Second, by Theorem 4.3, any accumulation point x^T of Lehrer's R_2 repeated allocation process, at stage T, satisfies $P\left[x^T \in \text{core}\left(\tilde{c}_R^T, \varepsilon_T\right)\right] = 1$.

Then, we prove that for any accumulation point x of a sequence $\{x^T\}_{T \ge 1}$ such that $x^T \in \text{core}\left(\tilde{c}_R^T, \varepsilon_T\right)$ a.s. for all $T > T'$, one has that x $\in \text{core}(Y)$ almost surely.

Since any $\tilde{c}_R^T(S) \xrightarrow[T\to\infty]{a.s.} Y_S$ for all S. Thus, for any $\delta > 0$, small enough, there exists $T(\delta)$ such that for all $T > T(\delta)$: $\tilde{c}_R^T(S) < Y_S + \delta$ almost surely. Now, because $x^T \in core(\tilde{c}_R^T, \varepsilon_T)$ almost surely, it follows that:

$$\begin{aligned} &x^T(S) \le \tilde{c}_R^T(S) + \varepsilon_T < Y_S + \delta + \varepsilon_T,\ \text{a.s.},\\ &\phi_T \ge \left|x^T(N) - E\left(\tilde{c}_R^T(N)\right)\right|, \end{aligned}$$

for all $T > T(\delta)$ and $S \subset N$.

Hence, as $\delta \to 0$ we have that $T(\delta) \to \infty$ and the accumulation point x satisfies:

$$\mathrm{x}(S) \le Y_S \text{ for all } S \subset N,\ \text{and } |\mathrm{x}(N) - E(Y_N)| \le \phi_Y, \tag{14}$$

almost surely since $\varepsilon_T(S) \xrightarrow[T\to\infty]{a.s.} 0$ for all $S \subseteq N$ and $\phi_T \underset{T\to\infty}{\to} \phi_Y$.

Any accumulation point of the sequence $\left\{\bar{a}^{R_2}\left(\tilde{c}_R^T\right)(T)\right\}_{T\ge1}$ belongs to core $\left(\tilde{c}_R^T, \varepsilon_T\right)$ almost surely for all $T > T'$. Hence, applying (14) any accumulation point of $\left\{\bar{a}^{R_2}\left(\tilde{c}_R^T\right)(T)\right\}$ belongs to core(Y) almost surely.

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